

Integral control of stable nonlinear systems

George Weiss and Vivek Natarajan

Abstract—Let \mathbf{P} be a nonlinear system described by $\dot{x} = f(x, u)$, $y = g(x)$, where the state trajectory x takes values in \mathbb{R}^n , u and y are scalar and f, g are of class C^1 . We assume that there is a Lipschitz function $\Xi: [u_{\min}, u_{\max}] \rightarrow \mathbb{R}^n$ such that for every constant input $u_0 \in [u_{\min}, u_{\max}]$, $\Xi(u_0)$ is an exponentially stable equilibrium point of \mathbf{P} . We also assume that $G(u) = g(\Xi(u))$, which is the steady state input-output map of \mathbf{P} , is strictly increasing. Denoting $y_{\min} = G(u_{\min})$ and $y_{\max} = G(u_{\max})$, we assume that the reference value r is in (y_{\min}, y_{\max}) . Our aim is that y should track r , i.e., $y \rightarrow r$ as $t \rightarrow \infty$, while the input of \mathbf{P} is only allowed to be in $[u_{\min}, u_{\max}]$. For this, we introduce a variation of the integrator, called the saturating integrator, and connect it in feedback with \mathbf{P} in the standard way, with gain $k > 0$. We show that for any small enough k , the closed-loop system is (locally) exponentially stable around an equilibrium point $(\Xi(u_r), u_r)$, with a “large” region of attraction $X_T \subset \mathbb{R}^n \times [u_{\min}, u_{\max}]$. When the state $(x(t), u(t))$ of the closed-loop system converges to $(\Xi(u_r), u_r)$, then the tracking error $r - y$ tends to zero. The compact set X_T can be made larger by choosing a larger parameter $T > 0$, but this may force us to use a smaller k , in which case the response of the system will be slower. Every initial state $(x_0, u_0) \in \mathbb{R}^n \times [u_{\min}, u_{\max}]$ such that the state trajectory of \mathbf{P} starting from x_0 , with constant input u_0 , converges to $\Xi(u_0)$, is contained in some set X_T for large enough T . If the open-loop system is globally asymptotically stable, then for every compact subset \mathcal{K} of the state space there exists a $k > 0$ such that all the closed-loop state trajectories starting from \mathcal{K} will converge to the unique equilibrium point.

I. INTRODUCTION AND THE DEFINITION OF THE SATURATING INTEGRATOR

In this short paper we prove some results about the integral control of stable nonlinear systems. Let the nonlinear time-invariant system \mathbf{P} be described by

$$\dot{x} = f(x, u), \quad y = g(x), \quad (1.1)$$

where f and g are C^1 functions. The state of this system is $x \in \mathbb{R}^n$, the input u and the output y are scalar. We assume that for each constant input function u_0 in a certain range $[u_{\min}, u_{\max}]$, \mathbf{P} has a locally exponentially stable equilibrium point $\Xi(u_0)$ and the function $\Xi: [u_{\min}, u_{\max}] \rightarrow \mathbb{R}^n$ is Lipschitz continuous. We are not allowed to apply to \mathbf{P} an input function with values outside the range $[u_{\min}, u_{\max}]$, either because the system may become unstable, or because of actuator saturation, or because of safety considerations (such as overvoltage on components) - the reason for this limitation is not relevant for the theory developed here. More technical assumptions will be stated in the later sections, here we want to explain the idea.

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G. Weiss (gweiss@eng.tau.ac.il) is with the School of Electrical Engineering, Tel Aviv University, Ramat Aviv, Israel, 69978, Ph: +972 36405313.

V. Natarajan (vivekn@sc.iitb.ac.in) is with the Systems and Control Engineering Group, Indian Institute of Technology (IIT) Bombay, Mumbai 400076, India, Ph: + 91 2225765385. He was formerly with Tel Aviv University.

It is intuitively appealing to regard \mathbf{P} as being approximately modelled by the memoryless system $y = g(\Xi(u))$, and this would be close to correct if u were a very slowly changing signal with values in the range $[u_{\min}, u_{\max}]$. We assume that the function $G = g \circ \Xi$ is strictly increasing and we denote

$$y_{\min} = G(u_{\min}), \quad y_{\max} = G(u_{\max}).$$

The control objective is to make y track a constant (but not given a-priori) reference signal $r \in (y_{\min}, y_{\max})$, while not allowing the input signal to exit the range $[u_{\min}, u_{\max}]$. If \mathbf{P} is replaced with the memoryless model $y = G(u)$ mentioned above, then this control objective can be achieved using an integral controller with saturation: for some $k > 0$,

$$\dot{v}(t) = k[r - y(t)], \quad u(t) = \text{sat}(v(t)),$$

where sat denotes a saturation function that does not allow u to exit the range $[u_{\min}, u_{\max}]$, and $u = v$ if v is inside the allowed range. It is not difficult to show that (for the memoryless model) this would work, i.e., the closed-loop system would be stable and we would have $y(t) \rightarrow r$ as $t \rightarrow \infty$.

The above very simple result (for the memoryless model) can be shown using a quadratic Lyapunov function, or it may be regarded as an application of the famous circle criterion, for which we refer to the nice survey [3]. Even for this situation, the saturation as described is not satisfactory, because during a fault the state u of the integrator may reach a very large value (a phenomenon called “windup”), from which it would take a long time to recover after the fault. A better way to build the integrator is to prevent its state from exiting the range $[u_{\min}, u_{\max}]$. There are different ways to do this, and such controllers are said to have *anti-windup*. There is a rich literature on control with anti-windup, with a much wider meaning for the concept, see for instance [6], [14], [15]. We propose one very particular controller with anti-windup, which we call the *saturating integrator*, a dynamical system defined by

$$\dot{u} = \mathcal{S}(u, w), \quad (1.2)$$

where

$$\mathcal{S}(u, w) = \begin{cases} w^+ & \text{if } u \leq u_{\min}, \\ w & \text{if } u \in (u_{\min}, u_{\max}), \\ w^- & \text{if } u \geq u_{\max}. \end{cases} \quad (1.3)$$

Here w^+ is the positive part of w and w^- is the negative part of w :

$$w^+ = \max\{w, 0\}, \quad w^- = \min\{w, 0\}.$$

The state of the saturating integrator is u and its state space is the interval $[u_{\min}, u_{\max}]$.

If w is a continuous function with finitely many zeros in every finite interval, then it is easy to define the corresponding state trajectories of the saturating integrator, even though the function \mathcal{S} is not continuous. However, if the zeros of w have

an accumulation point, then the definition of state trajectories u of this system may become problematic. For instance, if $u(0) = u_{\max}$ and $w(t) = t \sin(1/t)$, then it is not obvious what the function u is. To overcome this problem, let us first consider only inputs w that are not problematic, for instance, polynomials. It is easy to check that if u_1 and u_2 are state trajectories of the saturating integrator corresponding to the polynomials w_1 and w_2 , respectively (and any initial states), and at some moment $t \geq 0$ we have $u_2(t) \geq u_1(t)$, then

$$\frac{d}{dt}[u_2(t) - u_1(t)] \leq |w_2(t) - w_1(t)|,$$

which implies that

$$\frac{d}{dt}|u_2(t) - u_1(t)| \leq |w_2(t) - w_1(t)|,$$

and by a symmetric argument this last inequality is true also when $u_2(t) < u_1(t)$. It follows that

$$|u_2(t) - u_1(t)| \leq |u_2(0) - u_1(0)| + \int_0^t |w_2(\sigma) - w_1(\sigma)| d\sigma.$$

This shows that $u(t)$ from (1.2) depends Lipschitz continuously both on $u(0)$ and also on w considered with the L^1 norm. Indeed, for $u_2(0) = u_1(0)$ we can write the last estimate as

$$|u_2(t) - u_1(t)| \leq \|w_2 - w_1\|_{L^1[0,t]}. \quad (1.4)$$

Hence, by continuous extension, we can define $u(t)$ for any input $w \in L^1[0,t]$ (because the polynomials are dense in $L^1[0,t]$). In block diagrams (such as Figure 1) we use the symbol $\int \mathcal{S}$ to denote the saturating integrator. The saturating integrator has been used also in [7].

The main results of this paper concern the feedback system shown in Figure 1, which is described by (1.1), (1.2) and $w = k(r - y)$. The state of the closed-loop system is $(x(t), u(t))$ and its state space is

$$X = \mathbb{R}^n \times [u_{\min}, u_{\max}]. \quad (1.5)$$

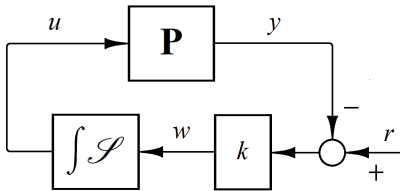


Figure 1. The closed-loop system formed from the plant \mathbf{P} , the saturating integrator $\int \mathcal{S}$ and the constant gain $k > 0$, with the constant reference r .

An informal statement of the main result of this paper is that with the saturating integrator as the controller in the feedback loop, under reasonable assumptions on the plant, for any constant reference r in the range (y_{\min}, y_{\max}) , the following holds: For any small enough feedback gain $k > 0$, the closed-loop system shown in Figure 1 is locally asymptotically stable around an equilibrium point, with a “large” region of attraction. When the state converges to this equilibrium point, then the tracking error $r - y$ tends to zero.

The precise statement of the main results and their proof will be given in Sections III and IV. This theory has been developed with a very specific example in mind: the control of the virtual

field current in a synchronverter. Explaining the context of that application would take several pages and instead we just refer to the papers [9], [10]. The material in this paper was originally meant to be a lemma in [10], but then it grew too long. The authors believe that the results are relevant for many more applications and they are amenable to various generalizations.

Our main results are related to those in [2], where \mathbf{P} is assumed to be built from a stable linear system connected in cascade with nondecreasing nonlinear functions (memoryless systems) both at its input and at its output. It seems that for such \mathbf{P} , our Theorem 3.4 follows from Theorem 7 and Remark 8 in [2]. Another class of related results concerns the situation when \mathbf{P} is assumed to be impedance passive, which allows an entirely different approach to the proof of set point regulation with closed-loop stability, with arbitrary positive gain, see for instance [11], [12], [4].

II. NONLINEAR SYSTEMS WITH SLOWLY VARYING INPUTS

In this section we investigate the behaviour of a nonlinear system \mathbf{P} from (1.1) (with f and g of class C^1), assuming that it has certain stability properties formulated in Assumption 1 below. It is well-known that for any initial state $x(0)$ and any continuous input function u , the differential equation in (1.1) has a unique solution defined on some maximal interval $[0, t^*)$ (possibly $t^* = \infty$), see [13, Appendix C] or [5, Chapter 3] for good discussions of this topic. It is important to note that if t^* is finite, then $\limsup_{t \rightarrow t^*} \|x(t)\| = \infty$, see for instance Exercise 3.26 in [5] (see also Corollary 2.3 in [4]). In this case, we say that the state trajectory has a *finite escape time* t^* . The two lemmas in this section imply that certain state trajectories of \mathbf{P} remain bounded as long as they exist, and this of course implies that they exist for all $t \geq 0$. In many places, our arguments should contain phrases like “if the solution exists for this t , then ...”. However, in order to make this text less clumsy, we will discuss about these state trajectories as if it is clear from the start that they exist for all $t \geq 0$.

Notation. For any interval J , any $\alpha > 0$ and any $m \in \mathbb{N}$, we denote by $\text{Lip}_\alpha(J; \mathbb{R}^m)$ the set of those $u: J \rightarrow \mathbb{R}^m$ which are Lipschitz continuous with Lipschitz constant α . If u is defined on a larger set containing J , then $u \in \text{Lip}_\alpha(J; \mathbb{R}^m)$ means that the restriction of u to J is in $\text{Lip}_\alpha(J; \mathbb{R}^m)$.

Assumption 1. There exist real numbers $u_{\min} < u_{\max}$, $\alpha > 0$ and a function $\Xi \in \text{Lip}_\alpha([u_{\min}, u_{\max}]; \mathbb{R}^n)$ such that

$$f(\Xi(u), u) = 0 \quad \forall u \in [u_{\min}, u_{\max}],$$

i.e., for each $u_0 \in [u_{\min}, u_{\max}]$, $\Xi(u_0)$ is an equilibrium point that corresponds to the constant input u_0 .

Moreover, \mathbf{P} is *uniformly exponentially stable* around these equilibrium points. This means that there exist $\varepsilon_0 > 0$, $\lambda > 0$ and $m \geq 1$ such that for each constant input function $u_0 \in [u_{\min}, u_{\max}]$, the following holds:

If $\|x(0) - \Xi(u_0)\| \leq \varepsilon_0$, then for every $t \geq 0$,

$$\|x(t) - \Xi(u_0)\| \leq m e^{-\lambda t} \|x(0) - \Xi(u_0)\|. \quad (2.1)$$

Remark 2.1: The uniform exponential stability condition above can be checked by linearization: If the Jacobian matrices

$$A(u_0) = \left. \frac{\partial f(x, u)}{\partial x} \right|_{\substack{x=\Xi(u_0) \\ u=u_0}} \in \mathbb{R}^{n \times n}$$

have eigenvalues bounded away from the right half-plane,

$$\max \operatorname{Re} \sigma(A(u_0)) \leq \lambda_0 < 0 \quad \forall u_0 \in [u_{\min}, u_{\max}],$$

then \mathbf{P} is uniformly exponentially stable, see (11.16) in [5]. Under Assumption 1, $\max \operatorname{Re} \sigma(A(u_0))$ is a continuous function of u_0 . Hence, if this function is always negative, then by the compactness of $[u_{\min}, u_{\max}]$, its maximum is also negative.. Thus, for the uniform exponential stability we only have to check that each of the matrices $A(u_0)$ is stable.

The following two lemmas show that under the above assumption, if the input u changes sufficiently slowly and stays in the relevant range of values $[u_{\min}, u_{\max}]$, and if $x(0)$ is close to its momentary equilibrium value $\Xi(u(0))$, then for all times $t > 0$, $x(t)$ remains close to $\Xi(u(t))$. These results are related to those in Section 9.3 of [5], where the proof technique is to construct and use Lyapunov functions. It is difficult to see the precise relationship between our lemmas and those in the cited reference, because the results depend on a lot of constant parameters. As far as the authors can see, the implication (2.2) below (which we need later) cannot be derived directly from the material in the cited reference.

Lemma 2.2: Assume that \mathbf{P} satisfies Assumption 1.

Then there exists $\kappa > 0$ and $T > 0$ such that for every $\varepsilon \in [0, \varepsilon_0]$ and every $u \in \operatorname{Lip}_{\kappa \varepsilon}([0, \infty); \mathbb{R})$ with $u(t) \in [u_{\min}, u_{\max}]$ for all $t \geq 0$, the following holds: for all $t \geq T$,

$$\|x(0) - \Xi(u(0))\| \leq \varepsilon \implies \|x(t) - \Xi(u(t))\| \leq \frac{2}{3}\varepsilon. \quad (2.2)$$

Proof. For $\varepsilon = 0$ the statement is clearly true, no matter how we choose κ and T . Thus, in the sequel we only consider $\varepsilon \in (0, \varepsilon_0]$. We consider $\delta > 0$ (to be specified later) and $u \in \operatorname{Lip}_{\delta}([0, \infty); \mathbb{R})$ with $u(t) \in [u_{\min}, u_{\max}]$ for all $t \geq 0$. We choose

$$T \geq \frac{1}{\lambda} \log [6m(m+1)]$$

and introduce the step function u_T which is obtained by sampling u with the sampling period T and holding the result constant between consecutive sampling moments:

$$u_T(t) = u((k-1)T) \quad \forall t \in [(k-1)T, kT), \quad k \in \mathbb{N}.$$

We note the following very simple properties of u_T :

- (i) $u_T(t) \in [u_{\min}, u_{\max}]$ for all $t \geq 0$.
- (ii) $|u_T(t) - u(t)| \leq \delta T$ for all $t \geq 0$.
- (iii) $\|\Xi(u_T(t)) - \Xi(u(t))\| \leq \alpha \delta T$ for all $t \geq 0$.

Suppose that for some $k \in \mathbb{N}$ we have

$$\|x((k-1)T) - \Xi(u((k-1)T))\| \leq \varepsilon. \quad (2.3)$$

Our current aim is to show that for suitable δ this implies

$$\|x(kT) - \Xi(u(kT))\| \leq \frac{\varepsilon}{2(m+1)}. \quad (2.4)$$

Let z_k be the state trajectory of \mathbf{P} with the input u_T and the initial condition $z_k((k-1)T) = x((k-1)T)$ (thus, $\dot{z}_k = f(z_k, u_T)$). According to (2.1), for all $t \in [(k-1)T, kT]$ we have

$$\|z_k(t) - \Xi(u((k-1)T))\| \leq m e^{-\lambda(t-(k-1)T)} \varepsilon. \quad (2.5)$$

In particular,

$$\|z_k(kT) - \Xi(u((k-1)T))\| \leq m e^{-\lambda T} \varepsilon \leq \frac{\varepsilon}{6(m+1)}. \quad (2.6)$$

From property (iii) above, taking limits as $t \rightarrow kT$, and assuming that $\alpha \delta T \leq \varepsilon/6(m+1)$, we have that

$$\|\Xi(u((k-1)T)) - \Xi(u(kT))\| \leq \frac{\varepsilon}{6(m+1)}.$$

Combining the last estimate with (2.6) (with the triangle inequality), we obtain that

$$\alpha \delta T \leq \frac{\varepsilon}{6(m+1)} \implies \|z_k(kT) - \Xi(u(kT))\| \leq \frac{\varepsilon}{3(m+1)}. \quad (2.7)$$

For any $\xi \in \mathbb{R}^n$ and $t \geq 0$, denote

$$\Delta f(\xi, t) = f(\xi, u_T(t)) - f(\xi, u(t)),$$

so that Δf is a C^1 function of ξ and it is piecewise continuous in t . We introduce the tubular open set

$$W = \left\{ \xi \in \mathbb{R}^n \mid \min_{u_0 \in [u_{\min}, u_{\max}]} \|\xi - \Xi(u_0)\| < \left(m + \frac{1}{6}\right) \varepsilon_0 \right\}.$$

Let L_2 be the Lipschitz constant of f with respect to its second argument u , when the first argument x is in W and $u \in [u_{\min}, u_{\max}]$. (L_2 is finite because f is a C^1 function in (x, u) and W is bounded.) Then it follows (using property (ii)) that for any $(\xi, t) \in W \times [0, \infty)$ we have

$$\|\Delta f(\xi, t)\| \leq L_2 \|u_T(t) - u(t)\| \leq L_2 \delta T.$$

We can now apply a result about the continuous dependence of the solutions on “the right-hand side” of the differential equation, stated as Theorem 3.4 in [5] (see also Theorem 55 in [13]). (What we denote by $f(x, u(t))$ is denoted in [5] by $f(t, x)$, what we denote by $(\Delta f)(\xi, t)$ is denoted in [5] by $g(t, \xi)$ and our number $L_2 \delta T$ is denoted in [5] by μ .) To apply the result from [5] on a time interval $[(k-1)T, \tau]$, where $\tau \in ((k-1)T, kT]$, we must check that both $z_k(t)$ and $x(t)$ remain in W for all $t \in [(k-1)T, \tau]$. For $z_k(t)$ this follows from (2.5), no matter what $\tau \in ((k-1)T, kT]$ is. For $x(t)$ at first we can only say that some possible values of τ exist, because $x((k-1)T) \in W$. We denote by τ^* the largest number $\tau \in ((k-1)T, kT]$ such that $x(t) \in W$ for all $t \in [(k-1)T, \tau]$ (this is simply the supremum of all the possible τ). From the result in [5] we obtain that for any $t \in [(k-1)T, \tau^*]$,

$$\|z_k(t) - x(t)\| \leq \frac{L_2 \delta T}{L_1} \left[e^{L_1(t-(k-1)T)} - 1 \right], \quad (2.8)$$

where L_1 is the Lipschitz constant of f with respect to its first argument x on the set W , when the second argument u varies over $[u_{\min}, u_{\max}]$ (L_1 is finite, since f is a C^1 function and W is bounded). Combining (2.5) with (2.8), we obtain that if

$$\frac{L_2 \delta T}{L_1} (e^{L_1 T} - 1) \leq \frac{\varepsilon}{6(m+1)}, \quad (2.9)$$

then

$$\|x(\tau^*) - \Xi(u((k-1)T))\| < \left(m + \frac{1}{6(m+1)}\right) \varepsilon. \quad (2.10)$$

This shows that $x(\tau^*) \in W$, and if $\tau^* < kT$ then this contradicts the maximality of τ^* . Hence, under the condition in (2.9), $\tau^* = kT$ and (2.8) implies that

$$\|z_k(kT) - x(kT)\| \leq \frac{\varepsilon}{6(m+1)}.$$

Combining the above with (2.7), we obtain that if

$$\alpha\delta T \leq \frac{\varepsilon}{6(m+1)} \quad \text{and} \quad \frac{L_2\delta T}{L_1}(e^{L_1T} - 1) \leq \frac{\varepsilon}{6(m+1)}, \quad (2.11)$$

then (2.4) holds. It is clear that both conditions in (2.11) can be satisfied by choosing δ sufficiently small. More precisely, what we need is that $\delta \leq \kappa\varepsilon$, where

$$\kappa = \min \left\{ \frac{1}{6(m+1)\alpha T}, \frac{L_1}{6(m+1)L_2T}(e^{L_1T} - 1)^{-1} \right\}.$$

Recall that our starting assumption in this segment of proof was (2.3), and for $\delta \leq \kappa\varepsilon$ we have obtained (2.4).

Now let us assume that the left side of (2.2) holds, which is (2.3) for $k = 1$. Then by induction it follows that for any input $u \in \text{Lip}_{\kappa\varepsilon}([0, \infty); \mathbb{R})$ (with κ as defined above),

$$\|x(kT) - \Xi(u(kT))\| \leq \frac{\varepsilon}{2(m+1)} \quad \forall k \in \mathbb{N}. \quad (2.12)$$

It remains to look at the values of $\|x(t) - \Xi(u(t))\|$ at the times $t \geq T$. From (2.8) we see that, assuming (2.11),

$$\|x(t) - z_k(t)\| \leq \frac{\varepsilon}{6(m+1)} \leq \frac{\varepsilon}{12} \quad \forall t \geq 0. \quad (2.13)$$

From (2.1) and (2.12) we have that for all $t \in [(k-1)T, kT]$ with $k \geq 2$,

$$\|z_k(t) - \Xi(u((k-1)T))\| \leq \frac{m\varepsilon}{2(m+1)} < \frac{\varepsilon}{2}. \quad (2.14)$$

From property (iii) at the beginning of this proof, using the first estimate in (2.11), we get that for all $t \in [(k-1)T, kT]$,

$$\|\Xi(u((k-1)T)) - \Xi(u(t))\| \leq \frac{\varepsilon}{6(m+1)} \leq \frac{\varepsilon}{12}. \quad (2.15)$$

Combining (2.13), (2.14) and (2.15), we get that if $u \in \text{Lip}_{\kappa\varepsilon}([0, \infty); \mathbb{R})$, then the conclusion in (2.2) holds. \square

Lemma 2.3: Suppose that Assumption 1 holds and let κ, T be the positive constants whose existence was proved in Lemma 2.2. Let $\varepsilon \in (0, \varepsilon_0]$ and assume that the initial state $x(0)$ and the input u of \mathbf{P} satisfy

$$\|x(0) - \Xi(u(0))\| \leq \varepsilon, \quad u \in \text{Lip}_{\kappa\varepsilon}([0, \infty); \mathbb{R})$$

and $u(t) \in [u_{\min}, u_{\max}]$ for all $t \geq 0$. Then

$$\|x(t) - \Xi(u(t))\| < \left(m + \frac{1}{6}\right)\varepsilon \quad \forall t \geq 0. \quad (2.16)$$

Proof. For $t \geq T$ this follows from the better estimate given in Lemma 2.2. For $t \in [0, T]$ we use the estimate (2.10) (with t in place of τ^*) to conclude that

$$\|x(t) - \Xi(u(0))\| < \left(m + \frac{1}{6(m+1)}\right)\varepsilon. \quad (2.17)$$

It follows from property (iii) in the proof of Lemma 2.2, together with (2.11), that

$$\|\Xi(u(0)) - \Xi(u(t))\| \leq \frac{1}{6(m+1)}\varepsilon.$$

Combining this last estimate with (2.17) (and using that $m \geq 1$), we easily obtain (2.16). \square

III. THE CLOSED-LOOP SYSTEM

In this section we discuss the behaviour of the closed-loop system from Figure 1, with the state space X from (1.5). There is an interesting and somewhat unclear connection between the lemmas in this section and Tikhonov's theorem concerning singularly perturbed systems of differential equations, see Theorem 11.1 (and also Theorem 11.2) in [5]. Our system may be regarded as a variation of a subclass of the systems studied in the cited theorems. However, Tikhonov's theorem concerns the asymptotic behaviour of the solutions on a parameter, which in our case is k , as $k \rightarrow 0$, and this is different from our concerns (we want to establish stability for any fixed k in a range). We do not think that any of our results can be obtained from Tikhonov's theorem.

We start with a proposition about local existence and uniqueness of state trajectories, which is not an obvious fact due to the discontinuity of \mathcal{S} . Note that in this proposition we do not impose Assumption 1 on \mathbf{P} .

Proposition 3.1: Let \mathbf{P} be described by (1.1) with f and g of class C^1 and let $\int \mathcal{S}$ be the saturating integrator as in (1.2) and (1.3). For every $x_0 \in \mathbb{R}^n$, every $u_0 \in [u_{\min}, u_{\max}]$, every $k \geq 0$ and every $r \in \mathbb{R}$ there exists a $\tau > 0$ such that the closed-loop system from Figure 1 has a unique state trajectory (x, u) defined on $[0, \tau]$, such that $x(0) = x_0$ and $u(0) = u_0$.

If τ is maximal (i.e., the state trajectory cannot be continued beyond τ) then $\limsup_{t \rightarrow \tau} \|x(t)\| = \infty$.

Proof. Let $R > 0$ and let B_R be the closed ball of radius R around x_0 in \mathbb{R}^n . Denote $M = \max\{\|f(x, u)\| \mid x \in B_R, u \in [u_{\min}, u_{\max}]\}$. Then it is clear from the mean value theorem that for any input function u with values in $[u_{\min}, u_{\max}]$, the state trajectory of \mathbf{P} exists and remains in B_R for all $t \leq R/M$. For any $\tau \in (0, R/M]$ we denote by C_τ the set of all the continuous functions on the interval $[0, \tau]$, with values in $[u_{\min}, u_{\max}]$. This is a complete metric space with the distance induced by the supremum norm of continuous functions. For x_0 fixed, we denote by T_τ the (nonlinear) operator determined by \mathbf{P} , that maps any input function $u \in C_\tau$ into an output function $y \in C[0, \tau]$. The operator T_τ is Lipschitz continuous, by a similar argument to the one we used to derive (2.8) (now we use B_R in place of W). The Lipschitz bound of T_τ , which we denote by L_T , can be chosen to be independent of τ .

For u_0 fixed, let us denote by S_τ the input to output map of the saturating integrator on the time interval $[0, \tau]$. The estimate (1.4) shows that S_τ is Lipschitz continuous, with the Lipschitz bound τ . If (x, u) is a state trajectory of the closed-loop system which is defined on $[0, \tau]$, then we must have

$$u = S_\tau(r - T_\tau u).$$

This can be regarded as a fixed point equation on C_τ . For τ sufficiently small so that $L_\tau \cdot \tau < 1$, the above equation has a unique solution according to the Banach fixed point theorem, see for instance [1, Sect. 3]. It is easy to see that if u is a solution of the fixed point equation and x is the corresponding state trajectory of \mathbf{P} starting from x_0 , then (x, u) is the desired state trajectory of the closed-loop system on $[0, \tau]$. The τ that we have just found is surely not maximal, because if

the solution exists on the closed interval $[0, \tau]$, then we can repeat the same argument starting from τ , and we get a larger interval of existence of the state trajectory.

To show that if $\tau > 0$ is maximal then the solution must blow up at τ , we can use the same technique that is used for differential equations with continuous dependence on the state, as cited at the beginning of Section II. \square

Assumption 2. The system \mathbf{P} satisfies Assumption 1 and moreover, the function

$$G(u) = g(\Xi(u)), \quad u \in [u_{\min}, u_{\max}]$$

satisfies the following: There exists $\mu > 0$ such that for any $u_1, u_2 \in [u_{\min}, u_{\max}]$ with $u_1 > u_2$,

$$G(u_1) - G(u_2) \geq 2\mu(u_1 - u_2). \quad (3.1)$$

(If G is differentiable then this is equivalent to $G' \geq 2\mu$.)

Notation. Recall from Section I that we denote $y_{\min} = G(u_{\min})$ and $y_{\max} = G(u_{\max})$, so that clearly $y_{\min} < y_{\max}$. For any $r \in (y_{\min}, y_{\max})$ we define $u_r = G^{-1}(r)$ and we define $\mathcal{G}_r : [u_{\min} - u_r, u_{\max} - u_r] \rightarrow \mathbb{R}$ by shifting the graph of G :

$$\mathcal{G}_r(v) = G(v + u_r) - r,$$

so that \mathcal{G}_r is an increasing Lipschitz function and $\mathcal{G}_r(0) = 0$. It is clear that (3.1) holds with \mathcal{G}_r in place of G .

Lemma 3.2: Consider the closed-loop system from Figure 1, where \mathbf{P} satisfies Assumption 2, $k > 0$ and $r \in (y_{\min}, y_{\max})$. Assume that $u(0) \in [u_{\min}, u_{\max}]$ and let $x(0) \in \mathbb{R}^n$ and $\tau, \eta^* > 0$ be such that the closed-loop state trajectory (x, u) exists for $t \in [0, \tau]$ (and possibly also later) and

$$|y(t) - G(u(t))| \leq \eta^* \quad \forall t \in [0, \tau]. \quad (3.2)$$

Then for all $t \in [0, \tau]$ we have

$$|G(u(t)) - r| \leq \max \left\{ \left| \mathcal{G}_r(e^{-\mu kt}(u(0) - u_r)) \right|, 2\eta^* \right\}. \quad (3.3)$$

Proof. Denote $\eta = y - G(u)$ and define

$$V(u) = \frac{1}{2}(u - u_r)^2 \quad \forall u \in \mathbb{R}.$$

(If \mathbf{P} would be replaced by the memoryless system $y = G(u)$, then V would be a Lyapunov function for the closed-loop system.) We claim that for each $t \in [0, \tau]$,

$$|G(u(t)) - r| > 2\eta^* \Rightarrow \dot{V}(u(t)) \leq -2\mu k V(u(t)). \quad (3.4)$$

To prove (3.4), first notice that from the definition of the saturating integrator it follows that if $u(0) \in [u_{\min}, u_{\max}]$, then $u(t) \in [u_{\min}, u_{\max}]$ for all $t \geq 0$, as long as the state trajectory is defined. As long as $u \in (u_{\min}, u_{\max})$, we have

$$\begin{aligned} \dot{V} &= (u - u_r)\dot{u} = (u - u_r)k(r - y) \\ &= -(u - u_r)k[G(u) - r + \eta] \end{aligned} \quad (3.5)$$

(we have dropped the notation for the dependence on t). If $|G(u(t)) - r| > 2\eta^*$, then it follows from (3.2) that $|\eta(t)| < \frac{1}{2}|G(u(t)) - r|$. Hence, $G(u(t)) - r + \eta(t)$ and $G(u(t)) - r$ have the same sign and $|G(u(t)) - r + \eta(t)| > \frac{1}{2}|G(u(t)) - r|$. Hence,

using that G is an increasing function, we have from (3.5) that if $|G(u(t)) - r| > 2\eta^*$ and $u \in (u_{\min}, u_{\max})$, then

$$\dot{V} = -|u - u_r| \cdot k \cdot |G(u) - r + \eta|$$

$$\leq -\frac{k}{2}|u - u_r| \cdot |G(u) - r| \leq -2\mu k V,$$

in accordance with (3.4). If $|G(u(t)) - r| > 2\eta^*$ and $u(t) = u_{\min}$, then $G(u) - r < 0$ (due to the assumption that $r \in (y_{\min}, y_{\max})$). Since, as explained a little earlier, $G(u) - r + \eta$ and $G(u) - r$ have the same sign, the input to the saturating integrator is $r - y > 0$, so that again $\dot{u} = k(r - y)$, hence (3.4) is again true. Finally for $|G(u(t)) - r| > 2\eta^*$ and $u(t) = u_{\max}$, by a similar argument we again obtain that (3.4) is true.

To prove (3.3), notice that if $|G(u(t)) - r| \leq 2\eta^*$ for all $t \in [0, \tau]$, then the claim is trivially true. Thus, we look at the case when there exists $t \in [0, \tau]$ such that $|G(u(t)) - r| > 2\eta^*$. Then it follows from (3.4) that (at the moment t) V is decreasing, whence $|u - u_r|$ is decreasing, whence $|G(u) - r|$ is decreasing. It follows that if $t \in [0, \tau]$ is such that $|G(u(t)) - r| > 2\eta^*$, then the same is true for all the smaller values of $t \geq 0$ (otherwise, starting from a smaller value, $|G(u) - r|$ would have to be increasing to reach for the first time its value at t , which is impossible from (3.4)). Hence, the set of those $t \in [0, \tau]$ for which $|G(u(t)) - r| > 2\eta^*$ is an interval of the form $[0, \tau_1)$, where $\tau_1 \leq \tau$, and on this interval the function $|G(u) - r|$ is decreasing and V satisfies (3.4), which implies that

$$V(t) \leq e^{-2\mu kt} V(0) \quad \forall t \in [0, \tau_1),$$

whence

$$|u(t) - u_r| \leq e^{-\mu kt} |u(0) - u_r| \quad \forall t \in [0, \tau_1). \quad (3.6)$$

Notice that on the interval $[0, \tau_1)$ the functions $u - u_r$ and its image through \mathcal{G}_r , which is $G(u) - r$, cannot cross zero, and hence they have constant (and equal) sign.

Let us first consider the case when $u(0) - u_r > 0$. Then, in light of the comments we just made about sign, (3.6) becomes

$$0 < u(t) - u_r \leq e^{-\mu kt} (u(0) - u_r) \quad \forall t \in [0, \tau_1).$$

From here, applying \mathcal{G}_r we get that

$$0 < G(u(t)) - r \leq \mathcal{G}_r(e^{-\mu kt} (u(0) - u_r)) \quad \forall t \in [0, \tau_1).$$

From here, the claim (3.3) follows.

Let us now consider the case when $u(0) - u_r < 0$. Then, in light of the recent comments about sign, (3.6) becomes

$$0 > u(t) - u_r \geq e^{-\mu kt} (u(0) - u_r) \quad \forall t \in [0, \tau_1).$$

From here, applying \mathcal{G}_r we get that

$$0 > G(u(t)) - r \geq \mathcal{G}_r(e^{-\mu kt} (u(0) - u_r)) \quad \forall t \in [0, \tau_1).$$

From here, again the claim (3.3) follows. \square

Lemma 3.3: Consider the closed-loop system from Figure 1, where \mathbf{P} satisfies Assumption 2 and $r \in (y_{\min}, y_{\max})$. Recall the constant $\kappa > 0$ from Lemma 2.2. Let δ_g be a Lipschitz bound of g over the bounded region $W \subset \mathbb{R}^n$ introduced after (2.7). Choose $\lambda, k > 0$ such that

$$\tilde{\lambda} = 2\delta_g \left(m + \frac{1}{6} \right), \quad k < \frac{2\kappa}{\delta_g(6m+1)}. \quad (3.7)$$

Then there exists $\tau > 0$ with the following property: If $\varepsilon \in [0, \varepsilon_0]$, $u(0) \in [u_{\min}, u_{\max}]$ and

$$\|x(0) - \Xi(u(0))\| \leq \varepsilon, \quad |G(u(0)) - r| \leq \tilde{\lambda}\varepsilon, \quad (3.8)$$

then the state trajectory of the closed-loop system exists for all $t \geq 0$ and for all $t \geq \tau$ we have

$$\|x(t) - \Xi(u(t))\| \leq \frac{2}{3}\varepsilon, \quad |G(u(t)) - r| \leq \frac{2}{3}\tilde{\lambda}\varepsilon.$$

Proof. The statement is clearly true for $\varepsilon = 0$, so in the sequel we only consider $\varepsilon \in (0, \varepsilon_0]$. Assume that ε , $x(0)$ and $u(0)$ are given that satisfy (3.8). We have

$$\begin{aligned} |y(0) - r| &\leq |y(0) - G(u(0))| + |G(u(0)) - r| \\ &\leq |g(x(0)) - g(\Xi(u(0)))| + \tilde{\lambda}\varepsilon \\ &\leq \delta_g \|x(0) - \Xi(u(0))\| + \tilde{\lambda}\varepsilon \leq (\delta_g + \tilde{\lambda})\varepsilon. \end{aligned}$$

It follows from (3.7) (using $m \geq 1$) that

$$k \cdot (\delta_g + \tilde{\lambda}) < \frac{4m+8/3}{6m+1} \kappa < \kappa.$$

Combining this with the previous estimate, we get that

$$k \cdot |y(t) - r| < \kappa \varepsilon \quad (3.9)$$

holds for $t = 0$. Since the state trajectory (x, u) exist on some interval of positive length (according to Proposition 3.1) and by the continuity of y (as a function of t), (3.9) remains true for all t in an interval of positive length.

We claim that the state trajectory exists and (3.9) remains true for all $t \geq 0$. Suppose that this is not the case. Then let t^* be the largest positive number such that the state trajectory (x, u) exists and (3.9) holds for all $t \in [0, t^*)$. According to the definition of the saturating integrator, it follows that $u \in \text{Lip}_{\kappa\varepsilon}([0, t^*]; \mathbb{R})$ (in particular, u is defined also at t^*). According to Lemma 2.3 we have that the function x exists on $[0, t^*]$ and

$$\|x(t) - \Xi(u(t))\| < \left(m + \frac{1}{6}\right) \varepsilon \quad \forall t \in [0, t^*].$$

We denote $\eta(t) = y(t) - G(u(t))$, then the last estimate implies

$$|\eta(t)| < \delta_g \left(m + \frac{1}{6}\right) \varepsilon = \frac{\tilde{\lambda}\varepsilon}{2} \quad \forall t \in [0, t^*]. \quad (3.10)$$

Consider first the case when $G(u(0)) - r \geq 0$, then it follows that $u(0) - u_r = \mathcal{G}_r^{-1}(G(u(0)) - r) \leq \mathcal{G}_r^{-1}(\tilde{\lambda}\varepsilon)$. According to Lemma 3.2 with $\eta^* = \tilde{\lambda}\varepsilon/2$ we get that for all $t \in [0, t^*]$,

$$|G(u(t)) - r| \leq \max \left\{ \mathcal{G}_r(e^{-\mu kt} \mathcal{G}_r^{-1}(\tilde{\lambda}\varepsilon)), \tilde{\lambda}\varepsilon \right\} = \tilde{\lambda}\varepsilon.$$

Now consider the case when $G(u(0)) - r < 0$. Then (using that \mathcal{G}_r is an increasing function) we have that $u(0) - u_r = \mathcal{G}_r^{-1}(G(u(0)) - r) \geq \mathcal{G}_r^{-1}(-\tilde{\lambda}\varepsilon)$. According to Lemma 3.2 with $\eta^* = \tilde{\lambda}\varepsilon/2$ we get that for all $t \in [0, t^*]$,

$$|G(u(t)) - r| \leq \max \left\{ \left| \mathcal{G}_r(e^{-\mu kt} \mathcal{G}_r^{-1}(-\tilde{\lambda}\varepsilon)) \right|, \tilde{\lambda}\varepsilon \right\} = \tilde{\lambda}\varepsilon.$$

We have obtained that regardless of the sign of $G(u(0)) - r$,

$$|G(u(t)) - r| \leq \tilde{\lambda}\varepsilon \quad \forall t \in [0, t^*].$$

Using the above estimate and (3.10), we have

$$\begin{aligned} |y(t) - r| &\leq |y(t) - G(u(t))| + |G(u(t)) - r| \\ &\leq \frac{\tilde{\lambda}}{2}\varepsilon + \tilde{\lambda}\varepsilon = \frac{3\tilde{\lambda}}{2}\varepsilon \quad \forall t \in [0, t^*]. \end{aligned}$$

Using (3.7) we obtain that for all $t \in [0, t^*]$,

$$k \cdot |y(t) - r| < \frac{6(m+1/6)}{6m+1} \kappa \varepsilon = \kappa \varepsilon,$$

so that (3.9) holds for $t = t^*$.

Due to Proposition 3.1 the state trajectory (x, u) exists for some time after t^* . By the continuity of y , (3.9) remains true for some time after t^* , contradicting the maximality of t^* . We conclude that (x, u) is defined and (3.9) holds for all $t \geq 0$, and similarly for our recent estimate for $|G(u(t)) - r|$:

$$|G(u(t)) - r| \leq \tilde{\lambda}\varepsilon \quad \forall t \in [0, \infty). \quad (3.11)$$

Note that (3.9) implies that $u \in \text{Lip}_{\kappa\varepsilon}([0, \infty); \mathbb{R})$. We can now apply Lemma 2.2 to conclude that for some $T > 0$,

$$\|x(t) - \Xi(u(t))\| \leq \frac{2}{3}\varepsilon \quad \forall t \geq T.$$

Thus we have proved the first statement in the last line of the lemma, with $\tau = T$.

Applying the function g to the last estimate, and recalling the function $\eta = y - G(u)$, we obtain

$$|\eta(t)| \leq \frac{2}{3}\delta_g\varepsilon = \frac{1}{3} \cdot \frac{\tilde{\lambda}\varepsilon}{m + \frac{1}{6}} < \frac{\tilde{\lambda}\varepsilon}{3} \quad \forall t \geq T.$$

Note that for $t \geq T$ this replaces the estimate (3.10), but it is smaller by a factor of $2/3$.

First consider the case when $G(u(T)) - r \geq 0$. Then (3.11) implies that $u(T) - u_r = \mathcal{G}_r^{-1}(G(u(T)) - r) \leq \mathcal{G}_r^{-1}(\tilde{\lambda}\varepsilon)$. We apply Lemma 3.2, with $\eta^* = \tilde{\lambda}\varepsilon/3$, and we do a shift in time, so that our initial time is T . For all $t \geq 0$ we get

$$|G(u(T+t)) - r| \leq \max \left\{ \mathcal{G}_r(e^{-\mu kt} \mathcal{G}_r^{-1}(\tilde{\lambda}\varepsilon)), \frac{2\tilde{\lambda}\varepsilon}{3} \right\}.$$

Since \mathcal{G}_r is a Lipschitz function with Lipschitz bound $\delta_g\alpha$, we have for any v in its domain that

$$2\mu|v| \leq |\mathcal{G}_r(v)| \leq \delta_g\alpha|v|. \quad (3.12)$$

Combining this with the previous estimate, we get

$$\begin{aligned} |G(u(T+t)) - r| &\leq \max \left\{ \frac{\delta_g\alpha}{2\mu} e^{-\mu kt} \mathcal{G}_r(\mathcal{G}_r^{-1}(\tilde{\lambda}\varepsilon)), \frac{2\tilde{\lambda}\varepsilon}{3} \right\} \\ &= \max \left\{ \frac{\delta_g\alpha}{2\mu} e^{-\mu kt}, \frac{2}{3} \right\} \tilde{\lambda}\varepsilon. \end{aligned} \quad (3.13)$$

Now consider the case when $G(u(T)) - r < 0$. Then (3.11) implies that $u(T) - u_r = \mathcal{G}_r^{-1}(G(u(T)) - r) \geq \mathcal{G}_r^{-1}(-\tilde{\lambda}\varepsilon)$. We apply again Lemma 3.2, with $\eta^* = \tilde{\lambda}\varepsilon/3$, and with the initial time T , getting that for all $t \geq 0$,

$$|G(u(T+t)) - r| \leq \max \left\{ \left| \mathcal{G}_r(e^{-\mu kt} \mathcal{G}_r^{-1}(-\tilde{\lambda}\varepsilon)) \right|, \frac{2\tilde{\lambda}\varepsilon}{3} \right\}.$$

Combining this with (3.12) we get

$$\begin{aligned} |G(u(T+t)) - r| &\leq \max \left\{ \frac{\delta_g\alpha}{2\mu} e^{-\mu kt} |\mathcal{G}_r(\mathcal{G}_r^{-1}(-\tilde{\lambda}\varepsilon))|, \frac{2\tilde{\lambda}\varepsilon}{3} \right\} \\ &= \max \left\{ \frac{\delta_g\alpha}{2\mu} e^{-\mu kt}, \frac{2}{3} \right\} \tilde{\lambda}\varepsilon, \end{aligned} \quad (3.14)$$

which is the same conclusion as in (3.13). Hence, (3.14) holds regardless of the sign of $G(u(T)) - r$.

It follows that

$$|G(u(T+t)) - r| \leq \frac{2}{3}\tilde{\lambda}\varepsilon \quad \forall t \geq \frac{1}{\mu k} \log \frac{3\delta_g\alpha}{4\mu}.$$

Thus we have proved the second statement in the last line of the lemma, with

$$\tau = T + \frac{1}{\mu k} \log \frac{3\delta_g\alpha}{4\mu}.$$

Obviously this τ works for the first statement as well. \square

Theorem 3.4: We work under the assumptions of Lemma 3.3 up to and including (3.7). Then $(\Xi(u_r), u_r)$ is a locally asymptotically stable equilibrium point of the closed-loop system from Figure 1, with the state space X from (1.5).

If the initial state $(x(0), u(0)) \in X$ of the closed-loop system satisfies $\|x(0) - \Xi(u(0))\| \leq \varepsilon_0$, then

$$x(t) \rightarrow \Xi(u_r), \quad u(t) \rightarrow u_r, \quad y(t) \rightarrow r,$$

and this convergence is at an exponential rate.

Proof. We introduce the coordinate transformation

$$\mathcal{T} : X \rightarrow \mathbb{R}^n \times [y_{\min} - r, y_{\max} - r]$$

as follows:

$$\begin{bmatrix} \xi \\ w \end{bmatrix} = \mathcal{T} \left(\begin{bmatrix} x \\ u \end{bmatrix} \right) = \begin{bmatrix} x - \Xi(u) \\ G(u) - r \end{bmatrix}.$$

This transformation is invertible, its inverse is

$$\begin{bmatrix} x \\ u \end{bmatrix} = \mathcal{T}^{-1} \left(\begin{bmatrix} \xi \\ w \end{bmatrix} \right) = \begin{bmatrix} \xi + \Xi(\mathcal{G}_r^{-1}(w) + u_r) \\ \mathcal{G}_r^{-1}(w) + u_r \end{bmatrix}.$$

Both \mathcal{T} and \mathcal{T}^{-1} are Lipschitz. Note that in the new coordinates, the equilibrium point under discussion is $(0, 0)$.

Lemma 3.3 says that there exists a $\tau > 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$, if the initial state (in the new coordinates) is in the rectangular box $\|\xi(0)\| \leq \varepsilon$, $|w(0)| \leq \tilde{\lambda}\varepsilon$, then for all $t \geq \tau$ the state $(\xi(t), w(t))$ will be in a rectangular box that is $2/3$ times smaller. Clearly this implies that (in the new coordinates) the origin is a locally asymptotically stable equilibrium point. Moreover, the state converges to this equilibrium point at an exponential rate. Clearly the same conclusions hold for the equilibrium point $(\Xi(u_r), u_r)$ in the original coordinates.

Finally, suppose that the initial state satisfies $u(0) \in [u_{\min}, u_{\max}]$ and $\|x(0) - \Xi(u(0))\| \leq \varepsilon_0$. The Lipschitz bound of g , denoted δ_g , can be chosen as large as needed, so that $\tilde{\lambda}$ (given by (3.7)) becomes sufficiently large so that $|G(u(0)) - r| \leq \tilde{\lambda}\varepsilon_0$ holds. Then we can apply our earlier argument to conclude that $(x(t), u(t))$ converges to $(\Xi(u_r), u_r)$ at an exponential rate. Since $y(t) = g(x(t))$ and g is a C^1 function, it follows that $y(t)$ converges to $g(\Xi(u_r)) = G(u_r) = r$ at an exponential rate. \square

IV. FINDING A LARGE DOMAIN OF ATTRACTION

In this section we show that, under a well-posedness assumption for the closed-loop system from Figure 1, we can find a large domain of attraction for the asymptotically stable equilibrium point whose existence was proved in Theorem 3.4. The following assumption is stronger than the local well-posedness result in Proposition 3.1.

Assumption 3. There exists $k_0 > 0$ such that for any $k \in [0, k_0]$, the closed-loop system formed by \mathbf{P} and the saturating integrator, as shown in Figure 1, with any $r \in (y_{\min}, y_{\max})$, has a unique state trajectory in forward time on the interval $[0, \infty)$, for any initial state in X .

Moreover, at any time $t \geq 0$, the state $(x(t), u(t))$ depends continuously on the initial state $(x(0), u(0))$.

The above assumption is not trivial, because the differential equations describing the closed-loop system are not continuous (the discontinuity is in \mathcal{S}). It is worth noting that the saturating integrator is irreversible (in time) and hence the closed-loop system usually has no uniquely defined backwards (in time) state trajectories.

We remark that the discontinuity in \mathcal{S} could be eliminated by using a more complicated saturating integrator, where the discontinuities are “polished off” by using linear interpolation in place of the jumps present in the definition (1.3), when $u(t)$ lies in one of the two short segments $[u_{\min}, u_{\min} + \gamma]$ or $[u_{\max} - \gamma, u_{\max}]$ (where $\gamma > 0$). We see no practical benefit in using this replacement of \mathcal{S} .

Theorem 4.1: Assume that \mathbf{P} satisfies Assumption 2 and moreover, it has well-defined backwards state trajectories for all $t < 0$, corresponding to any initial state in \mathbb{R}^n and any constant input in $[u_{\min}, u_{\max}]$. Further, assume that the closed-loop system from Figure 1 satisfies Assumption 3, and $r \in (y_{\min}, y_{\max})$. Let $T > 0$ and define the set $X_T \subset X$ as follows: $(x_0, u_0) \in X$ belongs to X_T if the state trajectory z of \mathbf{P} starting from $z(0) = x_0$, with constant input u_0 , satisfies $\|z(T) - \Xi(u_0)\| \leq \varepsilon_0/2$.

Then there exists $k_T \in (0, k_0]$ such that for any $k \in (0, k_T]$, if the initial state of the closed-loop system is in X_T , then the state trajectory (x, u) of the closed-loop system satisfies

$$x(t) \rightarrow \Xi(u_r), \quad u(t) \rightarrow u_r, \quad y(t) \rightarrow r,$$

and this convergence is at an exponential rate.

Proof. Let $\tilde{X}_T \subset X$ consist of all the points in the state space X that a state trajectory of the closed-loop system can reach at some time $t \in [0, T]$, using any fixed value of $k \in [0, k_0]$ and starting from an initial state $(x_0, u_0) \in X_T$ at time 0. Obviously $X_T \subset \tilde{X}_T$. We claim that \tilde{X}_T is compact.

To prove this claim, first we note that X_T is compact. Indeed, the system \mathbf{P} together with a generator of constant inputs, together described by the differential equations

$$\dot{x}(t) = f(x(t), u(t)) \quad \dot{u}(t) = 0,$$

has well defined backward state trajectories, given by a continuous backward flow. The set X_T is the image of

$$\mathcal{M} = \left\{ (z_0, u_0) \in X \mid \|z_0 - \Xi(u_0)\| \leq \frac{\varepsilon_0}{2} \right\}$$

through the backward flow mentioned earlier, at time $-T$. It is easy to see that \mathcal{M} is compact, and hence X_T (its image through the backward flow) is also compact.

Now consider the system with state (x, u, k) and state space $X \times [0, k_0]$ defined by the equations

$$\dot{x}(t) = f(x(t), u(t)), \quad y(t) = g(x(t), u(t)),$$

$$\begin{aligned} \dot{u}(t) &= \mathcal{S}(u(t), k(t)[r - y(t)]), \quad \dot{k}(t) = 0, \\ x(0) &= x_0, \quad u(0) = u_0, \quad k(0) = k_0. \end{aligned}$$

In other words, this is just the usual closed-loop system, but we regard k as a constant state variable, that may also take the value 0 (which corresponds to constant u). From Assumption 3 we see that this system has a continuous semiflow

$$\Phi : X \times [0, k_0] \times [0, \infty) \rightarrow X \times [0, k_0],$$

so that $\Phi(x_0, u_0, k_0, t)$ is its state at time t . Notice that

$$\tilde{X}_T = \Pi\Phi(X_T \times [0, k_0] \times [0, T]),$$

where Π denotes projection onto the first component in the product $X \times [0, k_0]$. This implies that indeed \tilde{X}_T is compact.

Take $(x_0, u_0) \in X_T$ and let z be the state trajectory of \mathbf{P} starting from $z(0) = x_0$, with constant input u_0 , so that by assumption $\|z(T) - \Xi(u_0)\| \leq \varepsilon_0/2$. Let (x, u) be the state trajectory of the closed-loop system with some $k \in (0, k_0]$ (to be specified later) starting from (x_0, u_0) . By definition, we know that $x(t) \in \tilde{X}_T$ for all $t \in [0, T]$. We have $u \in \text{Lip}_\delta([0, T]; \mathbb{R})$ where (using the definition of the saturating integrator) the Lipschitz bound δ can be estimated as

$$\begin{aligned} \delta &= \max\{k|r - y(t)| \mid t \in [0, T]\} \\ &\leq k \max\{|r - g(\xi)| \mid (\xi, w) \in \tilde{X}_T\}. \end{aligned}$$

Using the same argument as in the derivation of (2.8), we obtain that

$$\|x(T) - z(T)\| \leq \frac{L_2 \delta T}{L_1} [e^{L_1 T} - 1],$$

where L_1 is the Lipschitz constant of f with respect to its first argument ξ , and L_2 is the Lipschitz constant of f with respect to its second argument w , when $(\xi, w) \in \tilde{X}_T$. Combining the last two estimates, we see that there exists a $p(T) > 0$ independent of the initial state in X_T such that $\|x(T) - z(T)\| \leq p(T) \cdot k$. Thus, we can choose $k_T^1 \in (0, k_0]$ small enough so that $\|x(T) - z(T)\| \leq \varepsilon_0/4$ for all $k \in [0, k_T^1]$. Combining this with $\|z(T) - \Xi(u_0)\| \leq \varepsilon_0/2$, we obtain that

$$\|x(T) - \Xi(u_0)\| \leq \frac{3\varepsilon_0}{4} \quad \forall k \in [0, k_T^1].$$

Finally, it is clear that $|u_0 - u(T)| \leq \delta T$, whence (remembering the constant α from Assumption 1) $\|\Xi(u_0) - \Xi(u(T))\| \leq \alpha \delta T$. Hence, we can find $k_T^2 \in (0, k_T^1]$ such that for $k \in [0, k_T^2]$ we have $\|\Xi(u_0) - \Xi(u(T))\| \leq \varepsilon_0/4$. Combining this with the previous estimate, we obtain that

$$\|x(T) - \Xi(u(T))\| \leq \varepsilon_0 \quad \forall k \in [0, k_T^2].$$

Now we can apply Theorem 3.4 (starting with the initial time T) to conclude that for any gain $k \in (0, k_T^2]$ which in addition satisfies (3.7), the functions x, u and y converge as stated. \square

Remark 4.2: The reason why we may call X_T a “large” domain of attraction is the following: If \mathbf{P} happens to be globally asymptotically stable (GAS) for every constant input $u_0 \in [u_{\min}, u_{\max}]$, then every initial state of the closed-loop system is contained in one of the sets X_T , if we choose T large enough. If we choose a “region of interest” $\mathcal{K} \subset X$ that is compact, then there exists a $k > 0$ such that all the closed-loop state trajectories starting from \mathcal{K} will converge to the unique equilibrium point. Indeed, the interiors of the sets X_T

are an open covering of \mathcal{K} , so that $\mathcal{K} \subset X_T$ if T is large enough. Then we have to choose a gain $k \leq k_T$. Of course, the price for choosing a very large T is that we may have to choose a very small gain k , and this may deteriorate the dynamic response of the closed-loop system.

Remark 4.3: Similar comments apply if \mathbf{P} is almost globally asymptotically stable (aGAS), as defined in [8], [9]. In the latter case, there may be a family $\Xi^j(u_0)$ ($j \in \mathbb{Z}$) of stable equilibrium points of \mathbf{P} corresponding to each constant input $u_0 \in [u_{\min}, u_{\max}]$, and our main results apply to each such family. Under the assumptions that we have seen earlier, for a given r , each such family of equilibrium points gives rise to one stable equilibrium point for the closed-loop system, which has its own domains of attraction X_T^j (for each $j \in \mathbb{Z}$ and $T > 0$). The union of all the sets X_T^j then covers almost all of X , according to the definition of the aGAS property.

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